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On isochronous Bruschi–Ragnisco–Ruijsenaars–Toda lattices: equilibrium configurations, behaviour in their neighbourhood, diophantine relations and conjectures

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Abstract

Isochronous versions of the Bruschi–Ragnisco–Ruijsenaars–Toda lattice and of some of its, also integrable, variants are introduced, their equilibrium configurations are found (when they exist), and by investigating the motions of these systems near equilibrium some *diophantine* relations are obtained as well as some insight into the solution of those of these integrable models whose solutions are not yet known.

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1. Introduction

The equations of motion of the Bruschi–Ragnisco–Ruijsenaars–Toda (BRRT) lattice read as follows:

$$\zeta_n'' = -\zeta_n' \left(\frac{\zeta_{n-1}'}{\zeta_n - \zeta_{n-1}} + \frac{\zeta_{n+1}'}{\zeta_n - \zeta_{n+1}} \right). \tag{1}$$

Here the N coordinates $\zeta_n(\tau)$ are the dependent variables, τ is the independent variable and appended primes denote differentiations (the reason for using Greek letters here for the dependent and independent variables, and appended primes rather than superimposed dots to denote differentiations, will be clear soon). Here and hereafter indices such as n, m run from 1 to N; but these equations of motion must be complemented by boundary conditions specifying their versions for the *extreme* values of the index $n, n = 0$ respectively $n = N$, when on the right-hand side of the equations of motion as written above comes into play the extra variables ζ_0 and ζ_{N+1}. Hereafter we consider the two standard prescriptions that maintain the *integrability*, indeed *solvability*, of the system of ODEs as written above: the 'periodic' assignment,

$$\zeta_0(\tau) = \zeta_N(\tau), \qquad \zeta_{N+1}(\tau) = \zeta_1(\tau), \tag{2a}$$

and respectively the 'free ends' assignment entailing that, on the right-hand side of (1) with $n = 1$ and $n = N$, the terms featuring ζ_0 or ζ_{N+1} must be omitted, a prescription that can clearly be implemented by setting

$$\zeta_0'(\tau) = \zeta_{N+1}'(\tau) = 0. \tag{2b}$$

The justification for associating the name of Toda with this system comes from its nearest-neighbour character, and it would be more evident if one were to perform the change of dependent variables $\zeta_n = \exp(q_n)$ (see, for instance, [1]). The justification for associating the name of Ruijsenaars with this system is that it belongs to the RS class of N-body problems (where the letter R can stand either for Ruijsenaars or for 'relativistic' [2]), see in particular [3]; indeed the equations of motion (1) with (2*b*) are exhibited in [2] in the context of the discussion of RS systems (see in particular pp 132-3 and equation (24) of section 2.1.13 of this book). And the justification for associating the names of Bruschi and Ragnisco with these models is that they were the first to solve them [4]. These systems were also discussed more recently as special solvable cases of the 'goldfish' model [5]. In that context an *isochronous* version (but less general than that considered below, therefore possessing no equilibrium configuration) was also treated.

In this paper, we concentrate on the *isochronous* version of the BRRT model, that is obtained by applying to the equations of motion (1) the standard trick (see, for instance, [5–22]) yielding *isochronous* systems, i.e. the change of (*independent* and *dependent*) variables

$$z_n(t) = \exp(\mathrm{i}\lambda\omega t)\zeta_n(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}. \tag{3}$$

Thereby the equations of motion (1) become

$$\ddot{z}_n - (2\lambda + 1)\,\mathrm{i}\omega\dot{z}_n - \lambda(\lambda + 1)\omega^2 z_n = -(\dot{z}_n - \mathrm{i}\lambda\omega z_n)\left(\frac{\dot{z}_{n-1} - \mathrm{i}\lambda\omega z_{n-1}}{z_n - z_{n-1}} + \frac{\dot{z}_{n+1} - \mathrm{i}\lambda\omega z_{n+1}}{z_n - z_{n+1}}\right). \tag{4}$$

Here and hereafter superimposed dots indicate differentiations with respect to the (*real*) independent variable t ('time'), and the N dependent variables $z_n(t)$ are the *complex* coordinates of N-point particles moving in the *complex* z-plane. Again these equations of motion must be complemented by boundary conditions, and we consider again the two cases that correspond to those singled out above, see (2*a*) and (2*b*): the 'periodic' case with the assignment,

$$z_0(t) = z_N(t), \qquad z_{N+1}(t) = z_1(t), \tag{5a}$$

and the *free ends* case, entailing again the disappearance of the terms featuring z_0 or z_{N+1}, which can be formally implemented via the assignment

$$\dot{z}_0(t) = z_0(t) = 0, \qquad \dot{z}_{N+1}(t) = z_{N+1}(t) = 0. \tag{5b}$$

These equations of motion, (4), reduce to (1) for $\omega = 0$ (in which case $\tau = t$, $z_n(t) = \zeta_n(\tau)$: see (3)), and this constant ω could be rescaled away when it does not vanish; but we prefer to keep it in evidence, and we hereafter assume it to be *positive*, $\omega > 0$, whenever we consider these *isochronous* systems. This constant ω sets the timescale, and we associate with it the basic period

$$T = \frac{2\pi}{\omega}. \tag{6}$$

As explained in detail elsewhere (see for instance [21, 22]) the motivation for introducing and investigating, whenever possible, *autonomous* 'ω-modified' systems obtained via the trick (3) is because clearly this transformation with λ *real* and *rational* (say, $\lambda = r/s$ with r and s coprime *integers* and $s > 0$) entails that, to every function $\zeta_n(\tau)$ that is *free of branch points* in the circular disc D of radius $1/\omega$ and centre i/ω in the *complex* τ-plane, there corresponds a function $z_n(t)$ that is *periodic* in t with period (at most) sT. If the functions $\zeta_n(\tau)$ are the solutions of an *integrable* system there is some justification to expect that all these functions are indeed free of branch points, and therefore that all the solutions of the corresponding 'ω-modified' model are *completely periodic*, entailing the *isochronous* character of this model. Let us, however, emphasize that one cannot be *certain* that the ω-modified system obtained in this manner from an *integrable* system is indeed *isochronous*, unless one is also able to show that *all* the solutions of the *integrable* model do have the property to be free of branch points in D (actually *isochronicity* also obtains if there is at most a *finite* number of branch points of *rational* exponent in D).

The *integrability* indeed *solvability* [4, 5] of the BRRT model (1) with (2a) and (2b) entail, via (3), that, if λ is a *rational* number (as we hereafter assume), the *generic* solution of the corresponding *isochronous* model (4) with (5) is *completely periodic* (thereby justifying the term *isochronous* attributed to this model: indeed perhaps a more appropriate terminology to denote this many-body system is to state that it describes an assembly of *nonlinear harmonic oscillators* [25]).

The main focus of this paper is to investigate the equilibrium configurations of this *isochronous* model (4) (with $\lambda \neq 0$: indeed in the special case with $\lambda = 0$, for both types of *boundary conditions*, *any* configuration with initially vanishing velocities, $\dot{z}_n(0)$, remains at rest throughout the time evolution; we do not discuss this special case in this paper). In the following section 2, we show that this system, (4), has *one* (or perhaps *several*, see below) equilibrium configurations (which we find explicitly) iff $\lambda = N$ in the *free ends* case, see (5b), while it has *no* equilibrium configuration at all in the *periodic* case, see (5a). In the subsequent section 3, we investigate the behaviour of this *isochronous* system in the *free ends* case, (4) with (5b), in the vicinity of its equilibrium configuration: the *isochronicity* of this system implies that the N periods of its small oscillations in that neighbourhood must *all* be *integer* multiples of the basic period T, and this yields a *diophantine* finding and leads us to proffer a *diophantine* conjecture. Finally in section 4, we extend these results to some other analogous models, which are as well known to be *integrable* but whose solutions are not explicitly known; and in this manner we are led to another *diophantine* conjecture, and also to some insight on the solutions of these *integrable* models. In particular, we are led to conjecture that the following system (clearly characterized by *free ends* boundary conditions) is as well *isochronous*:

$$\ddot{z}_1 + \mathrm{i}(2N-1)\omega\dot{z}_1 - N(N-1)\omega^2 z_1 = \frac{(\dot{z}_1 - \mathrm{i}\lambda\omega z_1)^2}{z_1 - z_2}, \tag{7a}$$

$$\ddot{z}_n + \mathrm{i}(2N-1)\omega\dot{z}_n - N(N-1)\omega^2 z_n = (\dot{z}_n - \mathrm{i}\lambda\omega z_n)^2 \left[\frac{1}{z_n - z_{n-1}} + \frac{1}{z_n - z_{n+1}}\right], \quad n = 2, \ldots, N-1, \tag{7b}$$

$$\ddot{z}_N + \mathrm{i}(2N-1)\omega\dot{z}_N - N(N-1)\omega^2 z_N = \frac{(\dot{z}_N - \mathrm{i}\lambda\omega z_N)^2}{z_N - z_{N-1}}. \tag{7c}$$

Note, the similarity yet difference of this model from (4) with (5b) and $\lambda = -N$.

Let us end this introductory section by mentioning that it is possible [26, 2, 5] to identify the *complex* plane in which the coordinates ζ_n move, as well as their counterparts z_n in the *isochronous* cases, with the *real* horizontal plane, thereby attributing a more physical significance to the corresponding equations of motion, which can then be interpreted as the *real rotation-invariant* equations of motions of Newtonian type describing N particles that move in the horizontal plane under the influence of certain one-body and two-body velocity-dependent forces. For instance the equations of motion (4) can be re-written as follows:

$$\ddot{\vec{r}}_n = (2\lambda + 1)\omega\hat{k} \wedge \dot{\vec{r}}_n + \lambda(\lambda + 1)\omega^2\vec{r}_n - r_{n,n-1}^{-2}[\vec{v}_n(\vec{v}_{n-1} \cdot \vec{r}_{n,n-1}) + \vec{v}_{n-1}(\vec{v}_n \cdot \vec{r}_{n,n-1}) - \vec{r}_{n,n-1}(\vec{v}_n \cdot \vec{v}_{n-1})] - r_{n,n+1}^{-2}[\vec{v}_n(\vec{v}_{n+1} \cdot \vec{r}_{n,n+1}) + \vec{v}_{n+1}(\vec{v}_n \cdot \vec{r}_{n,n+1}) - \vec{r}_{n,n+1}(\vec{v}_n \cdot \vec{v}_{n+1})], \tag{8a}$$

$$\vec{v}_n \equiv \dot{\vec{r}}_n - \mathrm{i}\lambda\omega\hat{k} \wedge \vec{r}_n. \tag{8b}$$

Here the *real* vector $\vec{r}_n(t) \equiv (\mathrm{Re}[z_n(t)], \mathrm{Im}[z_n(t)], 0)$ identifies the position of the nth particle in the horizontal plane, $\hat{k} = (0, 0, 1)$ is the unit vector orthogonal to the horizontal plane, $\vec{r}_{n,m} \equiv \vec{r}_n - \vec{r}_m$ so that $r_{n,m}^2 = r_n^2 + r_m^2 - 2\vec{r}_n \cdot \vec{r}_m$, and the rest of the notation is, we trust, self-evident. Analogous considerations apply to all the models considered in this paper: we leave to the interested reader the task to display (if need be, see chapter 4 of the book [2]) the 'more physical' *real* versions of the Newtonian equations of motion which are exhibited above and below only in their *complex* avatars. This possibility provides an additional 'physical' motivation to investigate the many-body problems treated in this paper.

2. Equilibrium configurations

Let us characterize the equilibrium configuration of the system (4) as follows:

$$z_n = u_n \qquad \dot{z}_n = 0, \qquad n = 1, \ldots, N. \tag{9}$$

We only consider *genuine* equilibrium configurations, characterized by values of N numbers u_n that avoid any vanishing of the denominators on the right-hand side of the equations of motion (4), hence satisfy (in the *free ends* case) the inequalities

$$u_1 \neq u_2, \tag{10a}$$

$$u_n \neq u_{n\pm 1}, \qquad n = 2, \ldots, N - 1, \tag{10b}$$

$$u_N \neq u_{N-1}. \tag{10c}$$

Moreover, we restrict hereafter consideration to nonvanishing values of the equilibrium coordinates u_n,

$$u_n \neq 0, \qquad n = 1, \ldots, N, \tag{11}$$

since it is clear from the equations of motion (4) that a particle sitting still at $z_n = 0$ neither feels nor contributes any force and is therefore altogether ignorable: actually the presence of any such particle, say $z_\nu = 0$, de-couples the problem into two separate ones: one involving only the coordinates $z_n(t)$ with $n > \nu$ and the other involving only the coordinates $z_n(t)$ with $n < \nu$.

Then the equations of (no) motion (4) with (5b) yield the relations

$$-\lambda - 1 = \lambda \frac{u_2}{u_1 - u_2}, \tag{12a}$$

$$-\lambda - 1 = \lambda \left(\frac{u_{n-1}}{u_n - u_{n-1}} + \frac{u_{n+1}}{u_n - u_{n+1}} \right), \qquad n = 2, \ldots, N - 1, \tag{12b}$$

$$-\lambda - 1 = \lambda \frac{u_{N-1}}{u_N - u_{N-1}}. \tag{12c}$$

Let us now introduce the ratios

$$\alpha_n = \frac{u_n}{u_{n+1}}, \qquad n = 1, \ldots, N-1, \tag{13}$$

whereby the N-nonlinear algebraic equations (12) read (after a convenient rearrangement) as follows:

$$\frac{1}{1-\alpha_1} = \frac{1}{\lambda} + 1, \tag{14a}$$

$$\frac{1}{1-\alpha_n} = \frac{1}{1-\alpha_{n-1}} + \frac{1}{\lambda}, \qquad n = 2, \ldots, N-1, \tag{14b}$$

$$\alpha_{N-1} = 1 + \lambda. \tag{14c}$$

The recursion (14*b*) with the initial condition (14*a*) is easily solved:

$$\frac{1}{1-\alpha_n} = \frac{n}{\lambda} + b, \tag{15}$$

where b is an arbitrary constant. Then the first boundary condition (14*a*) entails $b = 1$, yielding

$$\alpha_n = \frac{n}{n+\lambda}, \qquad n = 1, \ldots, N-1. \tag{16}$$

And the second condition (14*c*) then determines λ uniquely

$$\lambda = -N, \tag{17}$$

entailing

$$\alpha_n = \frac{n}{n-N}, \qquad n = 1, \ldots, N-1. \tag{18}$$

From this last formula and (13) one also gets

$$u_n = (-)^{n-1} \frac{(N-1)!}{(n-1)!(N-n)!} u = (-)^{n-1} \binom{N-1}{n-1} u, \tag{19}$$

where $u = u_1$ is an arbitrary (nonvanishing) constant (whose presence reflects the scaling invariant character of the equations that determine the numbers u_n, see above).

We conclude that the equilibrium configuration of the *isochronous* system (4) with (5*b*) is *uniquely* determined by the first-particle position $u_1 = u$, that can be assigned arbitrarily (up to the condition (11)). Note that for *even* N the equilibrium positions u_n are all distinct, i.e. $u_n \neq u_m$ if $n \neq m$, while for *odd* N clearly $u_n = u_{N-n+1}$ (for *even* N clearly $u_n = -u_{N-n+1}$). Actually the following results do not require the determination of the equilibrium coordinates u_n: their ratios α_n suffice (see (13) and (18)).

Of course all these findings are consistent with the known solution [3, 2] of this N-body system, and indeed could have been retrieved from it.

The treatment in the case with 'periodic' boundary conditions, see (5*a*), is analogous. In this case the equations of (no) motion yield the relations

$$-\lambda - 1 = \lambda \left(\frac{u_2}{u_1 - u_2} + \frac{u_N}{u_1 - u_N} \right), \tag{20a}$$

$$-\lambda - 1 = \lambda \left(\frac{u_{n-1}}{u_n - u_{n-1}} + \frac{u_{n+1}}{u_n - u_{n+1}} \right), \qquad n = 2, \ldots, N-1, \tag{20b}$$

$$-\lambda - 1 = \lambda \left(\frac{u_{N-1}}{u_N - u_{N-1}} + \frac{u_1}{u_N - u_1} \right). \tag{20c}$$

Proceeding as above, see (13), we obtain again the recursion (14*b*) but now it is supplemented by the requirement that definition (13) holds also for $n = 0$ and for $n = N$ with the assignments

$$\alpha_0 = \alpha_N = \frac{u_N}{u_1}. \tag{21}$$

The requirement that solution (15) of the recursion relation (14*b*) holds for $n = 0$ then determines the constant b, and one obtains thereby the formula

$$\alpha_n = \frac{\lambda\alpha_0 + n(1-\alpha_0)}{\lambda + n(1-\alpha_0)}, \qquad n = 1, \ldots, N. \tag{22}$$

It is thereby immediately seen that condition (21) entails

$$\alpha_0 = 1, \tag{23}$$

implying via (22)

$$\alpha_n = 1, \qquad n = 1, \ldots, N.$$

This entails the unacceptable result that all the equilibrium positions coincide, $u_n = u_1$. We, therefore, conclude that in this case the N-body problem has *no* genuine equilibrium configuration.

3. Behaviour near equilibrium and diophantine relations

Let us now consider the behaviour of our *isochronous* system (4) with (5*b*) in the neighbourhood of its equilibrium configuration, as determined in the preceding section. To this end we set

$$z_n(t) = u_n + \epsilon w_n(t), \tag{24}$$

and we then insert this assignment in the equations of motion (4) with (5*b*) treating ϵ as a small parameter. We thus get the linearized equations of motion

$$\ddot{w}_1 - (2\lambda+1)\,\mathrm{i}\omega\dot{w}_1 - \lambda(\lambda+1)\omega^2 w_1 = \mathrm{i}\lambda\omega\left[\frac{u_2\dot{w}_1 + u_1\dot{w}_2}{u_1 - u_2}\right] - \lambda^2\omega^2\left[\frac{u_2^2 w_1 - u_1^2 w_2}{(u_1-u_2)^2}\right], \tag{25a}$$

$$\begin{aligned}\ddot{w}_n - (2\lambda+1)\,\mathrm{i}\omega\dot{w}_n - \lambda(\lambda+1)\omega^2 w_n &= \mathrm{i}\lambda\omega\left[\frac{u_{n-1}\dot{w}_n + u_n\dot{w}_{n-1}}{u_n - u_{n-1}} + \frac{u_{n+1}\dot{w}_n + u_n\dot{w}_{n+1}}{u_n - u_{n+1}}\right]\\ &\quad - \lambda^2\omega^2\left[\frac{u_{n-1}^2 w_n - u_n^2 w_{n-1}}{(u_n-u_{n-1})^2} + \frac{u_{n+1}^2 w_n - u_n^2 w_{n+1}}{(u_n-u_{n+1})^2}\right],\\ &\quad n = 2, \ldots, N-1,\end{aligned} \tag{25b}$$

$$\begin{aligned}\ddot{w}_N - (2\lambda+1)\,\mathrm{i}\omega\dot{w}_N - \lambda(\lambda+1)\omega^2 w_N &= \mathrm{i}\lambda\omega\left[\frac{u_{N-1}\dot{w}_N + u_N\dot{w}_{N-1}}{u_N - u_{N-1}}\right]\\ &\quad - \lambda^2\omega^2\left[\frac{u_{N-1}^2 w_N - u_N^2 w_{N-1}}{(u_N-u_{N-1})^2}\right],\end{aligned} \tag{25c}$$

namely,

$$\underline{\ddot{w}} + \mathrm{i}\omega\underline{A}\,\underline{\dot{w}} - \omega^2\underline{B}\,\underline{w} = 0. \tag{26}$$

Here and below, to underline the vector and matrix character of our formulae, N-vectors are denoted by *lower* case *underlined* letters, hence $\underline{w} = \underline{w}(t)$ denotes the N-vector of components

$w_n = w_n(t)$, and likewise $N \otimes N$ matrices are denoted by *upper* case *underlined* letters. In particular, the two (constant) matrices $\underline{A}$ and $\underline{B}$ are defined (componentwise) as follows (as implied by (25) with (13) and (18)):

$$A_{n,n} = N, \qquad A_{n,n-1} = N+1-n, \qquad A_{n,n+1} = n, \tag{27a}$$

$$B_{n,n} = -2n^2 + (N+1)(2n-1) \equiv \tfrac{1}{2}[N^2 - 1 - n^2 - (N+1-n)^2], \tag{27b}$$

$$B_{n,n-1} = (N+1-n)^2, \qquad B_{n,n+1} = n^2. \tag{27c}$$

Note the simple symmetry properties of these formulae under the transformation $n \mapsto N+1-n$.

The general solution of the system of linear ODEs (26) is provided by formula

$$\underline{w}(t) = \sum_{k=1}^{2N} a_k \exp(\mathrm{i} p_k \omega t) \underline{v}^{(k)}, \tag{28}$$

where the $2N$ constants a_k are arbitrary (to be determined, in the context of the initial-value problem, from the $2N$ initial data $w_n(0)$ and $\dot{w}_n(0)$), while the numbers p_k, respectively, the corresponding N vectors $\underline{v}^{(k)}$, are the $2N$ eigenvalues, respectively the $2N$ eigenvectors, of the following (generalized) N-vector eigenvalue equation:

$$p_k^2 \underline{v}^{(k)} + p_k \underline{A}\, \underline{v}^{(k)} + \underline{B}\, \underline{v}^{(k)} = 0, \qquad k = 1, \ldots, 2N. \tag{29}$$

Hence the numbers p_k are the $2N$ roots of the following equation (polynomial of degree $2N$) in p:

$$\det[p^2 \underline{\mathbf{1}} + p\underline{A} + \underline{B}] = 0. \tag{30}$$

Here and throughout $\underline{\mathbf{1}}$ denotes of course the $N \otimes N$ unit matrix, $(\underline{\mathbf{1}})_{nm} = \delta_{nm}$.

But we already know, from our previous treatment, that the solutions of the *isochronous* model (4) are *completely periodic* with period T, see (6). The same periodicity property must, therefore, characterize the behaviour of solution (28) describing the behaviour of the system in the neighbourhood of its equilibrium configuration. We thus arrive at the following *diophantine finding*: *the* $2N$ *roots* p_k *of the polynomial equation (30) with (27) are all integers*.

In fact, motivated by this finding and by some numerical checks, we make the following *diophantine*

Conjecture 3.1. *Let the two* $N \otimes N$ *tridiagonal matrices* $\underline{A}$ *and* $\underline{B}$ *be defined by (27), then*

$$\det[p^2 \underline{\mathbf{1}} + p\underline{A} + \underline{B}] = p(p+N) \prod_{k=1}^{N-1} (p+k)^2. \tag{31}$$

Examples of the (true) diophantine relations entailed, for increasing values of N, by this formula follow:

$$\begin{vmatrix} p^2 + 2p + 1 = (p+1)^2 & p+1 \\ p+1 & p^2 + 2p + 1 = (p+1)^2 \end{vmatrix} = p(p+1)^2(p+2), \tag{32a}$$

$$\begin{vmatrix} p^2 + 3p + 2 = (p+1)(p+2) & p+1 & 0 \\ 2p+4 & p^2 + 3p + 4 & 2p+4 \\ 0 & p+1 & p^2 + 3p + 2 = (p+1)(p+2) \end{vmatrix}$$
$$= p(p+1)^2(p+2)^2(p+3), \tag{32b}$$

$$\begin{vmatrix} p^2+4p+3=(p+1)(p+3) & p+1 & 0 & 0 \\ 3p+9 & p^2+4p+7 & 2p+4 & 0 \\ 0 & 2p+4 & p^2+4p+7 & 3p+9 \\ 0 & 0 & p+1 & p^2+4p+3=(p+1)(p+3) \end{vmatrix}$$
$$= p(p+1)^2(p+2)^2(p+3)^2(p+4). \tag{32c}$$

While we have no doubts about the validity of this conjecture, because of the way it was arrived at and the numerical checks we made, to actually *prove* it for *all* values of N one should *solve* the eigenvalue problem (29) with (27): this task is probably possible, but it does not appear to be quite trivial.

4. Extensions to two other analogous models

In this section we indicate to what extent the findings reported above can be extended to two other analogous models, and we thereby arrive at some other *diophantine* relations as well as to some interesting insights about the solutions of the second of these *integrable* models.

The first model we consider is characterized by the equations of motion

$$\ddot{z}_1 - \mathrm{i}\omega\dot{z}_1 - \lambda\omega^2 z_1 = \frac{\dot{z}_1^2}{z_1} - \frac{(\dot{z}_1 - \mathrm{i}\lambda\omega z_1)(\dot{z}_2 - \mathrm{i}\lambda\omega z_2)z_1}{(z_1 - z_2)z_2}, \tag{33a}$$

$$\ddot{z}_n - \mathrm{i}\omega\dot{z}_n - \lambda\omega^2 z_n = \frac{\dot{z}_n^2}{z_n} - (\dot{z}_n - \mathrm{i}\lambda\omega z_n)\cdot\left[\frac{\dot{z}_{n-1} - \mathrm{i}\lambda\omega z_{n-1}}{z_n - z_{n-1}} + \frac{(\dot{z}_{n+1} - \mathrm{i}\lambda\omega z_{n+1})z_n}{(z_n - z_{n+1})z_{n+1}}\right],$$
$$n = 2, \ldots, N-1, \tag{33b}$$

$$\ddot{z}_N - \mathrm{i}\omega\dot{z}_N - \lambda\omega^2 z_N = \frac{\dot{z}_N^2}{z_N} - (\dot{z}_N - \mathrm{i}\lambda\omega z_N)\left[\frac{\dot{z}_{N-1} - \mathrm{i}\lambda\omega z_{N-1}}{z_N - z_{N-1}}\right], \tag{33c}$$

that are obtained via the usual trick, see (3) (with an obvious change of notation), from the equations of motion (23) of section 2.1.13 of [2] (note that these equations of motion are, up to the change of variables $u_n(t) = c^{-2n}[2aq_n(t)]$, just those of the Ruijsenaars–Toda system, see equation (19) *ibidem*, for the *free ends* type of boundary conditions (2*b*)). Because of the way this model has been obtained it is presumably *isochronous*.

A treatment completely analogous to that performed above for the model (4) yields for the equilibrium positions u_n (see (9)) the relations

$$1 = -\lambda\frac{u_1}{u_1 - u_2}, \tag{34a}$$

$$1 = -\lambda\left[\frac{u_{n-1}}{u_n - u_{n-1}} + \frac{u_n}{u_n - u_{n+1}}\right], \qquad n = 2, \ldots, N-1, \tag{34b}$$

$$1 = -\lambda\frac{u_{N-1}}{u_N - u_{N-1}}, \tag{34c}$$

that we conveniently rewrite in terms of the ratios α_n (see (13)) as follows:

$$\frac{1}{1-\alpha_1} = 1 + \frac{1}{\lambda}, \tag{35a}$$

$$\frac{\alpha_n}{1-\alpha_n} - \frac{\alpha_{n-1}}{1-\alpha_{n-1}} = \frac{1}{\lambda}, \qquad n = 2, \ldots, N-1, \tag{35b}$$

$$\alpha_{N-1} = \frac{1}{1-\lambda}. \tag{35c}$$

The recursion (35*b*) with (35*a*) is easily solved

$$\alpha_n = \frac{n}{n+\lambda}, \qquad n = 1, \ldots, N-1, \tag{36}$$

and the insertion of this expression of α_n with $n = N-1$ in (35*c*) entails

$$\lambda = 0, \tag{37}$$

hence the unacceptable result

$$\alpha_n = 1. \tag{38}$$

We conclude that this N-body problem has *no* genuine equilibrium configuration.

An analogous treatment of the variant of this model (33*b*) with *periodic*, rather than *free ends*, boundary conditions (namely, the model with the equations of motion (33*b*) assumed valid for all values of n including $n = 1$ and $n = N$ and with (5*a*)) yields the same conclusion (the detailed derivation is left as an exercise for the diligent reader).

An analogous treatment is as well applicable to the 'ω-modified' systems, see below, that we obtain via the usual trick, see (3), from equation (2) of section 4.4.7 of [2] (see p 486 of this book; the following assignments and notational changes should be performed before applying the trick (3): $a = 0, b = 0, z_n(t) \mapsto \zeta_n(\tau)$; the explicit version of these equations correspond of course to the $\omega = 0$ case of the equations written below, (39)). As indicated above (see section 1), since the original system is known [30, 31] to be *integrable* (at least for some appropriate boundary conditions [27, 28]), one might expect (but cannot be certain, since the solutions of these systems are not known) that the 'ω-modified' systems obtained in this manner are *isochronous*. We shall see below that our treatment provides some insights in this respect.

When the system identified above is complemented with *free ends* boundary conditions the equations of motion of its 'ω-modified' version read as follows:

$$\ddot{z}_1 + \mathrm{i}(2\lambda c - 1)\omega\dot{z}_1 + \lambda(\lambda c - 1)\omega^2 z_1 = (1+c)\frac{\dot{z}_1^2}{z_1} - \frac{c(\dot{z}_1 - \mathrm{i}\lambda\omega z_1)^2}{z_1 - z_2}, \tag{39a}$$

$$\ddot{z}_n + \mathrm{i}(2\lambda c - 1)\omega\dot{z}_n + \lambda(\lambda c - 1)\omega^2 z_n = (1+c)\frac{\dot{z}_n^2}{z_n} - c(\dot{z}_n - \mathrm{i}\lambda\omega z_n)^2\left[\frac{1}{z_n - z_{n-1}} + \frac{1}{z_n - z_{n+1}}\right], \qquad n = 2, \ldots, N-1, \tag{39b}$$

$$\ddot{z}_N + \mathrm{i}(2\lambda c - 1)\omega\dot{z}_N + \lambda(\lambda c - 1)\omega^2 z_N = (1+c)\frac{\dot{z}_N^2}{z_N} - \frac{c(\dot{z}_N - \mathrm{i}\lambda\omega z_N)^2}{z_N - z_{N-1}}. \tag{39c}$$

An analogous model, associated with *periodic* boundary conditions, is characterized instead by the validity of the ODEs (39*b*) for *all* values of n (including $n = 0$ and $n = N$), with the additional prescription (5*a*) to make sense of these ODEs also when $n = 0$ or $n = N$. In this second case, a treatment completely analogous to that performed above leads again to the conclusion that there is *no* genuine equilibrium configuration.

In the case characterized by the system of ODEs (39) as written above (corresponding therefore to *free ends* boundary conditions) a treatment completely analogous to that performed above allows instead, as in the case treated in section 2, to identify an equilibrium configuration (again unique up to rescaling), with the quantities α_n (see (13)) defined now as follows:

$$\alpha_n = \frac{n-N}{n}, \tag{40}$$

and correspondingly with the two parameters λ and c related by the formula:

$$\lambda c = N. \tag{41}$$

Let us then proceed and analyse, following again the previous treatment (see section 3), the behaviour of this model, (39), in the neighbourhood of its equilibrium configuration. We thereby get the following linearized equations of motion (in the notation of section 3):

$$\ddot{\underline{w}} + \mathrm{i}\omega \underline{\tilde{A}}\, \dot{\underline{w}} - c^{-1}\omega^2 \underline{\tilde{B}}\, \underline{w} = 0, \tag{42}$$

with the two (constant) $N \otimes N$ matrices $\underline{\tilde{A}}$ and $\underline{\tilde{B}}$ defined now (componentwise) as follows:

$$\tilde{A}_{nm} = \delta_{nm}, \qquad \underline{\tilde{A}} = \underline{\mathbf{1}}, \tag{43}$$

$$\tilde{B}_{n,n} = N(N-1) - (n-1)^2 - (N-n)^2, \tag{44a}$$

$$\tilde{B}_{n,n-1} = (n-1)^2, \qquad \tilde{B}_{n,n+1} = (N-n)^2. \tag{44b}$$

Note that the matrix $\underline{\tilde{A}}$ is now *diagonal* (in fact it coincides with the unit matrix), while the matrix $\underline{\tilde{B}}$ is *tridiagonal*; and note the symmetrical character of this second matrix under the exchange $n-1 \mapsto N-n$.

The solution of (42) reads (in the notation of section 3; the second version of this formula obtains via (41))

$$\underline{w}(t) = \sum_{k=1}^{2N} a_k \exp\left(\frac{\mathrm{i}p_k\lambda\omega t}{N}\right) \underline{v}^{(k)} = \sum_{k=1}^{2N} a_k \exp\left(\frac{\mathrm{i}p_k\omega t}{c}\right) \underline{v}^{(k)}, \tag{45}$$

where the $2N$ numbers p_k are the $2N$ eigenvalues of the generalized eigenvalue equation

$$\left[\left(p_k^2 + cp_k\right)\underline{\mathbf{1}} + c\underline{\tilde{B}}\right]\underline{v}^{(k)} = 0, \tag{46}$$

hence they are the $2N$ roots of the following polynomial equation (of degree $2N$ in p):

$$\det[(p^2 + cp)\underline{\mathbf{1}} + c\underline{\tilde{B}}] = 0. \tag{47}$$

By setting $p^2 + cp = q$, one clearly sees that these $2N$ numbers p_k are given by the formula

$$p_m = \tfrac{1}{2}[-c + (c^2 - 4cq_m)^{1/2}], \qquad p_{N+m} = \tfrac{1}{2}[-c - (c^2 - 4cq_m)^{1/2}], \qquad m = 1, \ldots, N, \tag{48}$$

where the N numbers q_m are now the N roots of the polynomial equation (of degree N in q)

$$\det[q\underline{\mathbf{1}} - \underline{\tilde{B}}] = 0. \tag{49}$$

Note that the parameter c has now dropped out from this formula (see (44)).

If we knew for sure that the system (39) were *isochronous*, we could *assert* that the numbers p_k must *all* be *rational* whenever λ, hence as well c (see (41)), are *rational*. But clearly this cannot be the case for an arbitrary *rational* value of the parameter c (see (48) and (49)). This demonstrates that the original system, of which the system (39) is the 'ω-modified' variant, in spite of its property to be *integrable*, does *not* feature only solutions whose analytic structure as functions of the independent variable is sufficiently simple to guarantee the *isochronicity* of (39). Numerical checks for small values of N (see below) do however indicate that the numbers p_k are indeed *rational* (in fact they are all *integers*: this motivates *a posteriori* the form of our *ansatz* (45)) for the special value $c = -1$ (entailing $\lambda = -N$, see (41) as well as some marginal simplification of the system (39); see (7)). We are therefore led to conjecture that, in this special case, the model (7) is indeed *isochronous*, and to proffer the following *diophantine*

Conjecture 4.1. *the tridiagonal $N \otimes N$ matrix $\underline{M} = 4\underline{\tilde{B}}$,*

$$M_{n,n} = 4[N(N-1) - (n-1)^2 - (N-n)^2], \tag{50a}$$

$$M_{n,n-1} = 4(n-1)^2, \qquad M_{n,n+1} = 4(N-n)^2, \tag{50b}$$

has the N eigenvalues $m^2 - 1, m = 1, \ldots, N$, i.e.

$$\det[\mu - \underline{M}] = \prod_{m=1}^{N} [\mu + 4m(1-m)] \tag{51}$$

Examples:

$$\begin{vmatrix} \mu-4 & -4 \\ -4 & \mu-4 \end{vmatrix} = \mu(\mu-8), \tag{52a}$$

$$\begin{vmatrix} \mu-8 & -16 & 0 \\ -4 & \mu-16 & -4 \\ 0 & -16 & \mu-8 \end{vmatrix} = \mu(\mu-8)(\mu-24), \tag{52b}$$

$$\begin{vmatrix} \mu-12 & -36 & 0 & 0 \\ -4 & \mu-28 & -16 & 0 \\ 0 & -16 & \mu-28 & -4 \\ 0 & 0 & -36 & \mu-12 \end{vmatrix} = \mu(\mu-8)(\mu-24)(\mu-48), \tag{52c}$$

$$\begin{vmatrix} \mu-16 & -64 & 0 & 0 & 0 \\ -4 & \mu-40 & -36 & 0 & 0 \\ 0 & -16 & \mu-48 & -16 & 0 \\ 0 & 0 & -36 & \mu-40 & -4 \\ 0 & 0 & 0 & -64 & \mu-16 \end{vmatrix} = \mu(\mu-8)(\mu-24)(\mu-48)(\mu-80). \tag{52d}$$

Note that completely analogous considerations to those proffered at the end of the preceding section are as well applicable here.

5. Outlook

The conjecture entailed by our treatment that the model (39) with $\omega = 0$, while *integrable* [27, 28, 30, 31], possesses solutions all of which are *meromorphic* functions of the independent variable (considered as a complex variable) if $c = -1$ is consistent with the *exact* solution of this model for $N = 2$, which can be easily obtained. Unfortunately, even for $N = 2$, we have not been able so far to obtain the exact solution of this model with $c \neq -1$, although it can be shown by perturbation techniques that the solution of the $N = 2$ model with $c = -1 + \varepsilon$ and ε a small parameter is indeed *not meromorphic*, consistently with our conjecture. The final settling of this issue remains as an open problem.

Another model (or rather a class of models) to which this approach is applicable is characterized by the equations of motion

$$\zeta_n'' = \left(\zeta_n'^2 + c\zeta^k\right)\left(\frac{1}{\zeta_n - \zeta_{n+1}} + \frac{1}{\zeta_n - \zeta_{n-1}}\right) - \frac{kc}{2}\zeta_n^{k-1}, \tag{53}$$

with $k = 0, 1, 2, 3$ or 4. These models are presumably *integrable* [29–31], provided they are complemented by appropriate *boundary conditions*, such as (2*a*) and (2*b*), specifying the assignments of ζ_0 and ζ_{N+1} [27, 28] . One easily sees that via the change of dependent variables $\zeta_n \longmapsto 1/\zeta_n$ this model (53) goes into a completely analogous one up to the corresponding change $k \longmapsto 4 - k$ (we are indebted to Ravil Yamilov for this observation). Hence it is sufficient to restrict attention to the three models with $k = 0, 1, 2$. A rather natural research plan is to introduce via the trick (3) *autonomous* variants of these *integrable* models; it is easily seen that this can be done (with an appropriate assignment of λ) for $k = 0$ and $k = 1$. As explained above, one expects then that the two 'ω-modified' models obtained in this manner be *isochronous*. One can then determine the equilibrium configurations of these two models: again, they exist, and are easily found, only for the *free ends* type of boundary conditions. One can finally investigate the behaviour of these *isochronous* systems in the vicinity of their equilibrium configurations and thereby obtain some *diophantine* results. These findings will be reported in a separate paper [32].

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